SOME SETS IN DIGITAL TOPOLOGY

Dr. N. Vithya,
Department of Mathematics,
SNS College of Engineering, Coimbatore.
nvithya.r@gmail.com

Abstract
The notions of strong and weak forms of open sets and closed sets in the digital line and the digital plane have been used in digital image filtering techniques. The purpose of this paper is to characterize such sets in Digital topology with special reference to $b$-closed sets and $b$-open sets.

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1. Introduction
Kong and Kopperman [7] gave a topological approach to digital topology. Kong et.al. [8] studied the digital fundamental group and also established that on a strongly normal digital picture space, the discrete and continuous concepts are equivalent. Maki et.al. [9] investigated the digital line and operation approaches of $T_{1/2}$ spaces. Devi et.al. [6] studied the topological properties of wgp-closed sets in the digital plane. Punam K. Saha et.al. [10] studied that the basic parts of digital geometry can be generalized into sets of convex voxels. Thangavelu [13] discussed the properties of non-empty digital intervals $[a, b] \cap \mathbb{Z}$ and cardinalities of subspace topology on digital intervals are characterized. In this paper, we characterize nearly open sets in the Khalimsky topology.

2. Preliminaries

Definition 2.1
A subset $A$ of space $X$ is said to be
(i) regular open (Stone, 1937) if $A = \text{int}(\text{cl}(A))$ and regular closed if $A = \text{cl}(\text{int}(A))$.
(ii) $\alpha$-open (Njastad, 1965) if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ and $\alpha$-closed if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.
(iii) semi-open (Levine, 1963) if $A \subseteq \text{cl}(\text{int}(A))$ and semi-closed if $\text{int}(\text{cl}(A)) \subseteq A$.
(iv) pre-open (Mashhour, 1982) if $A \subseteq \text{int}(\text{cl}(A))$ and pre-closed if $\text{cl}(\text{int}(A)) \subseteq A$.
(v) semi-pre-open (Andrijivec, 1986) or $\beta$-open (Abd El-Monsef, 1983) if $A \subseteq \text{cl}(\text{int}(A))$ and semi-pre-closed or $\beta$-closed if $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$.
(vi) $\beta$-open (Andrijivec, 1996) if $A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$ and $b$-closed if $\text{cl}(\text{int}(A)) \cap \text{int}(\text{cl}(A)) \subseteq A$.
(vii) $\beta$-open (Indira, 2012) if $A \subseteq \text{cl}(\text{int}(A)) \cap \text{int}(\text{cl}(A))$ and $\beta$-closed if $\text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A)) \subseteq A$.
(viii) a p-set (Thangavelu et.al. 2002 (a)) if $\text{cl}(\text{int}(A)) \subseteq \text{int}(\text{cl}(A))$.
(ix) a q-set (Thangavelu et.al. 2002 (b)) if $\text{int}(\text{cl}(A)) \subseteq \text{cl}(\text{int}(A))$.
(x) $b^\#$-open (Usha Parameswari et.al. 2014) if $A = \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$ and $b^\#$-closed if $A = \text{cl}(\text{int}(A)) \cap \text{int}(\text{cl}(A))$.
(xi) $b^{**}$-open (Bharathi et.al. 2011) if $A \subseteq \text{int}(\text{cl}(\text{int}(A))) \cup \text{cl}(\text{int}(\text{cl}(A)))$ and $b^{**}$-closed if $\text{int}(\text{cl}(\text{int}(A))) \cap \text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.

Lemma 2.2 (Khalimsky et.al. 1990)
Let $A$ be a subset of $\mathbb{Z}$. Then
(i) $A$ is open if and only if for every $x$ in $A$ the following holds.
(x is odd) or (x is even with x-1, x+1 ∈ A).
(ii) A is closed if and only if for every x in A the following holds.
(x is even) or (x is odd with x-1, x+1 ∈ A).

Let N(x) denote the smallest neighbourhood of x in (Z, K). Then

\[
N(x) = \begin{cases} 
  \{x\} & \text{if } x \text{ is odd} \\
  \{x-1, x, x+1\} & \text{if } x \text{ is even}
\end{cases}
\]

Let N(x) denote the smallest closed set containing x in (Z, K). Then

\[
N(x) = \begin{cases} 
  \{x\} & \text{if } x \text{ is even} \\
  \{x-1, x, x+1\} & \text{if } x \text{ is odd}
\end{cases}
\]

3. Characterization of sets in Khalimsky topology

Proposition 3.1
Suppose if x is an odd integer then the set \{x\} is b*-closed, b**-open, b-clopen, regular open, semi-open, semi-pre-open, a q-set, a t*-set in (Z, K).

Proof
Suppose x is odd, \text{int}_K \{x\} = \{x\} and \text{cl}_K \text{int}_K \{x\} = \{x-1, x, x+1\}, \text{cl}_K \{x\} = \{x-1, x, x+1\} and \text{int}_K \text{cl}_K \{x\} = \{x\}. \text{int}_K \text{cl}_K \text{int}_K \{x\} = \{x\} and \text{cl}_K \text{int}_K \text{cl}_K \{x\} = \{x-1, x, x+1\}. Since \text{int}_K \text{cl}_K \{x\} \cap \text{cl}_K \text{int}_K \{x\} = \{x\}, by Definition 2.1(x), \{x\} is b*-closed and \text{int}_K \text{cl}_K \{x\} \cup \text{cl}_K \text{int}_K \{x\} = \{x-1, x, x+1\} by Definition 2.1(vi), \{x\} is b-open. Since every b*-closed set is b-closed, \{x\} is b-closed and hence \{x\} is b-clopen. Now \text{int}_K \text{cl}_K \text{int}_K \{x\} \cup \text{cl}_K \text{int}_K \text{cl}_K \{x\} = \{x-1, x, x+1\} \supseteq \{x\}, then by Definition 2.1(ix), \{x\} is b**-open. Since \{x\} = \text{int}_K \text{cl}_K \{x\} by Definition 2.1(i), \{x\} is regular open and \{x\} \subseteq \text{cl}_K \text{int}_K \{x\} = \{x-1, x, x+1\} by Definition 2.1(iii), \{x\} is semi-open. Also \{x\} \subseteq \text{cl}_K \text{int}_K \text{cl}_K \{x\} by Definition 2.1(v), \{x\} is semi-pre-open and \text{int}_K \text{cl}_K \{x\} \subseteq \text{cl}_K \text{int}_K \{x\} by Definition 2.1(iix), \{x\} is a q-set. \text{cl}_K \{x\} = \text{cl}_K \text{int}_K \{x\} implies \{x\} is a t*-set.

Proposition 3.2
Suppose if x is an even integer the set \{x\} is b-closed, b***-open, b***-closed, *b*-closed, a-open, a-closed, pre-closed, semi closed, a t-set in (Z, K).

Proof
Suppose x is even, \text{int}_K \{x\} = \emptyset and \text{cl}_K \text{int}_K \{x\} = \emptyset. Then \text{cl}_K \{x\} = \emptyset and \text{int}_K \text{cl}_K \{x\} = \emptyset. \text{int}_K \text{cl}_K \text{int}_K \{x\} = \{x-1, x, x+1\} and \text{cl}_K \text{int}_K \text{cl}_K \{x\} = \emptyset. Since \text{int}_K \text{cl}_K \{x\} \cap \text{cl}_K \text{int}_K \{x\} = \emptyset \subseteq \{x\} by Definition 2.1(vi), \{x\} is b-closed and \text{int}_K \text{cl}_K \{x\} \cup \text{cl}_K \text{int}_K \{x\} = \emptyset \subseteq \{x\} that implies \{x\} is *b*-closed. Since \text{int}_K \text{cl}_K \text{int}_K \{x\} \cup \text{cl}_K \text{int}_K \text{cl}_K \{x\} = \{x-1, x, x+1\} \supseteq \{x\} by Definition 2.1(xi), \{x\} is b**-open and \text{int}_K \text{cl}_K \{x\} \cap \text{cl}_K \text{int}_K \{x\} = \emptyset \subseteq \{x\} by Definition 2.1(xi), \{x\} is b**-closed. Also \{x\} \subseteq \text{int}_K \text{cl}_K \text{int}_K \{x\} by Definition 1.1(ii), \{x\} is a-open and \text{cl}_K \text{int}_K \text{cl}_K \{x\} \subseteq \{x\} by Definition 2.1(ii), \{x\} is a-closed. \text{cl}_K \text{int}_K \{x\} \subseteq \{x\} by Definition 2.1(iv), \{x\} is pre-closed and \text{int}_K \text{cl}_K \{x\} \subseteq \{x\} by Definition 2.1(iii) implies \{x\} is semi-closed. Also \text{int}_K \{x\} = \text{int}_K \text{cl}_K \{x\} implies \{x\} is a t-set and \text{cl}_K \{x\} = \text{cl}_K \text{int}_K \{x\} implies \{x\} is a t*-set.

Corollary 3.3
In (Z, K), the set \{x\} is b-clopen, b***-open and a t*-set for every positive integer x.

Proof
By Proposition 3.1, \{x\} is b-clopen, b***-open and a t*-set if x is odd and by Proposition 3.2, \{x\} is b-clopen, b***-open and a t*-set if x is even. This implies \{x\} is b-clopen, b***-open and a t*-set for every positive integer x.

Proposition 3.4
In (Z, K), \{x, x+1\} is b-clopen, b***-open, b***-closed, semi closed, semi open, semi-pre-open, semi-pre-closed, a q-set, a t-set, a t*-set for every positive integer x.
Proof
Let A = {x, x+1}. Suppose x is odd. \( \text{int}_k(A) = \{x\} \) and \( \text{cl}_k \text{int}_k(A) = \{x-1, x, x+1\} \). \( \text{cl}_k(A) = \{x-1, x, x+1\} \) and \( \text{int}_k \text{cl}_k(A) = \{x\} \). Then \( \text{int}_k \text{cl}_k \text{int}_k(A) = \{x\} \) and \( \text{cl}_k \text{int}_k \text{cl}_k(A) = \{x-1, x, x+1\} \). Since \( \text{int}_k \text{cl}_k(A) \cap \text{cl}_k \text{int}_k(A) = \{x\} \subseteq A \) by Definition 2.1.(vi), A is \( b^- \)-open and \( \text{int}_k \text{cl}_k(A) \cup \text{cl}_k \text{int}_k(A) = \{x-1, x, x+1, x+2\} \supseteq A \) implies A is \( b^- \)-closed. Thus A is b-clopen. Now \( \text{int}_k \text{cl}_k(A) \subseteq \text{cl}_k \text{int}_k(A) = \{x-1, x, x+1\} \supseteq A \), by Definition 2.1.(ix), A is \( b^- \)-open and \( \text{int}_k \text{cl}_k(A) \cap \text{cl}_k \text{int}_k(A) = \{x\} \subseteq A \) implies A is \( b^- \)-closed. Since \( \text{int}_k \text{cl}_k(A) = \{x\} \subseteq A \) implies by Definition 2.1.(iii), A is semi-closed and A \( \subseteq \text{cl}_k \text{int}_k(A) = \{x-1, x, x+1\} \) implies A is semi-open. Now A \( \subseteq \text{cl}_k \text{int}_k(A) = \{x-1, x, x+1\} \) by Definition 2.1.(v) implies A is semi-pre-open and \( \text{int}_k \text{cl}_k \text{int}_k(A) = \{x\} \subseteq A \) implies A is semi-pre-closed. Since \( \text{int}_k \text{cl}_k(A) = \{x\} \subseteq \text{cl}_k \text{int}_k(A) = \{x-1, x, x+1\} \) by Definition 2.1(ix), A is q-set and \( \text{int}_k(A) = \text{int}_k \text{cl}_k(A) \) implies A is a t-set. Also \( \text{cl}_k(A) = \text{cl}_k \text{int}_k(A) \) implies A is a t^-set.

Suppose x is even. \( \text{int}_k(A) = \{x+1\} \) and \( \text{cl}_k \text{int}_k(A) = \{x, x+1, x+2\} \). \( \text{cl}_k(A) = \{x, x+1, x+2\} \) and \( \text{int}_k \text{cl}_k(A) = \{x+1\} \). Then \( \text{int}_k \text{cl}_k \text{int}_k(A) = \{x+1\} \) and \( \text{cl}_k \text{int}_k \text{cl}_k(A) = \{x, x+1, x+2\} \). Since \( \text{int}_k \text{cl}_k(A) \cap \text{cl}_k \text{int}_k(A) = \{x+1\} \subseteq A \) by Definition 2.1.(vi), A is \( b^- \)-open and \( \text{int}_k \text{cl}_k(A) \cup \text{cl}_k \text{int}_k(A) = \{x, x+1, x+2\} \supseteq A \) implies A is \( b^- \)-closed. Also \( \text{int}_k \text{cl}_k(A) \cup \text{cl}_k \text{int}_k(A) = \{x, x+1, x+2\} \) implies A is a t-set. Therefore for every integer x, A is b-clopen, \( b^- \)-open, \( b^- \)-closed, semi open, semi open, semi-pre-open, semi-pre-closed, a q-set, a t-set, a t^-set.

Proposition 3.5
In \((\mathbb{Z}, k)\), the set \( \{x-1, x, x+1\} \) is \( b^- \)-open, b-clopen, \( b^- \)-closed, semi-closed, semi-pre-closed, a q-set, a t-set, a t^-set if x is an odd integer.

Proof
Let A = \{x-1, x, x+1\}. \( \text{int}_k(A) = \{x\} \) and \( \text{cl}_k \text{int}_k(A) = \{x-1, x, x+1\} \). \( \text{int}_k \text{cl}_k \text{int}_k(A) = \{x\} \) and \( \text{cl}_k \text{int}_k \text{cl}_k(A) = \{x-1, x, x+1\} \). Since \( \text{int}_k \text{cl}_k(A) \cap \text{cl}_k \text{int}_k(A) = \{x\} \subseteq \{x-1, x, x+1\} \) by Definition 2.1.(vi), A is b-closed. Now by Definition 2.1(x), \( \text{int}_k \text{cl}_k(A) \cup \text{cl}_k \text{int}_k(A) = A \) implies A is \( b^- \)-open and since every \( b^- \)-open set is b-open, A is b-open and therefore b-clopen. Since \( \text{int}_k \text{cl}_k \text{int}_k(A) \cup \text{cl}_k \text{int}_k \text{cl}_k(A) \subseteq A \) by Definition 2.1(x), A is \( b^- \)-closed and Definition 2.1.(iii), \( \text{int}_k \text{cl}_k(A) = \{x\} \subseteq A \) implies A is semi-closed. Now by Definition 2.1(v) \( \text{int}_k \text{cl}_k \text{int}_k(A) = \{x\} \subseteq A \) implies A is semi-pre-closed and by Definition 2.1(ix), \( \text{int}_k \text{cl}_k(A) = \{x\} \subseteq \text{cl}_k \text{int}_k(A) \) implies A is a q-set. Since \( \text{int}_k(A) = \text{int}_k \text{cl}_k(A) \) implies A is a t-set and \( \text{cl}_k(A) = \text{cl}_k \text{int}_k(A) \) implies A is a t^-set.

Proposition 3.6
In \((\mathbb{Z}, K)\), the set \( \{x-1, x, x+1\} \) is regular open, \( b^- \)-closed, b-clopen, \( b^- \)-open, semi-open, semi-pre-open, a q-set, a t-set, a t^-set if x is an even integer.

Proof
Let A = \{x-1, x, x+1\} and suppose x is even. \( \text{int}_k(A) = A \) and \( \text{cl}_k \text{int}_k(A) = \{x-2, x-1, x, x+1, x+2\} \). \( \text{int}_k \text{cl}_k \text{int}_k(A) = A \) and \( \text{cl}_k \text{cl}_k \text{int}_k(A) = \{x-2, x-1, x, x+1, x+2\} \). Since \( \text{A} = \text{int}_k \text{cl}_k(A) = \{x-1, x, x+1\} \), by Definition 2.1(i), A is regular open. Also by Definition 2.1(xi), \( \text{int}_k \text{cl}_k(A) \cap \text{cl}_k \text{int}_k(A) = A \) implies A is \( b^- \)-closed and Definition 2.1(vi), \( \text{int}_k \text{cl}_k(A) \cup \text{cl}_k \text{int}_k(A) = \{x-2, x-1, x, x+1, x+2\} \supseteq A \) that implies A is b-open. Since every \( b^- \)-closed set is b-closed, A is b-closed and therefore b-clopen. Since \( \text{int}_k \text{cl}_k \text{int}_k(A) \cup \text{cl}_k \text{int}_k \text{cl}_k(A) = \{x-2, x-1, x, x+1, x+2\} \supseteq A \)
by Definition 2.1(xi), A is b"-open and by Definition 2.1(iii), A ⊆ clK intk(A) = {x-2, x-1, x, x+1, x+2} implies A is semi-open. Now by Definition 2.1(v) A ⊆ clK intK clK(A) = {x-2, x-1, x, x+1, x+2} implies A is semi-pre-open and intK clK(A) = A ⊆ clK intK(A) = {x-2, x-1, x, x+1, x+2} by Definition 2.1(ix) implies that A is a q-set. Also intK(A) = intK clK(A) implies A is a t-set and clK(A) = clK intK(A) implies A is a t'-set.

**Corollary 3.7**
In (Z, K), the set {x-1, x, x+1} is b-clopen, a q-set, a t-set, a t'-set for every positive integer x.

**Proof**
Let A = {x-1, x, x+1}. Then by Proposition 3.5, A is b-clopen, q-set, t-set, t'-set if x is odd and by Proposition 3.6, A is b-clopen, a q-set, a t-set and a t'-set if x is even. This implies A is b-clopen, a q-set, a t-set and a t'-set for every positive integer x.

**Corollary 7.8**
In (Z, K), the set {x, x+1, x+2} is regular open, b*-closed, b-clopen, b"-open, semi-open, semi-pre-open, a q-set, a t-set, a t'-set and a t"-set if x is an odd integer.

**Proof**
If x is odd then {x+1} is even and by Proposition 3.6, {x, x+1, x+2} is regular open, b*-closed, b-clopen, b"-open, semi-open, semi-pre-open, a q-set, a t-set and a t'-set.

**Corollary 3.9**
In (Z, K), the set {x, x+1, x+2} is b*-open, b-clopen, b"-closed, semi-closed, semi-pre-closed, a q-set, a t-set and a t'-set if x is an even integer.

**Proof**
If {x} is even then {x+1} is odd by Proposition 3.5, {x, x+1, x+2} is b*-open, b-clopen, b"-closed, semi-closed, semi-pre-closed, a q-set, a t-set and a t'-set.

**Theorem 3.10**
In (Z, k), any three consecutive integers form either a b*-closed set or b*-open set.

**Proof**
Let A = {x, x+1, x+2}. If x is odd, then from Corollary 3.8, A is b*-closed and if x is odd, by Corollary 3.9, A is b*-open. This proves that any three consecutive integers form either a b*-closed set or b*-open set.

**Proposition 3.11**
In (Z, k), {x, x+1, x+2, x+3} is b-clopen, semi-closed, semi-closed, semi-pre-closed, b"-open, b"-closed, a q-set, a t-set and a t'-set for every positive integer x.

**Proof**
Let A = {x, x+1, x+2, x+3} and suppose x is odd. intK(A) = {x, x+1, x+2} and clK intK(A) = {x-1, x, x+1, x+2, x+3}. intK clK intK(A) = {x, x+1, x+2} and clK(A) = {x-1, x, x+1, x+2, x+3}. intK clK(A) = {x, x+1, x+2} and clK intK clK(A) = {x-1, x, x+1, x+2, x+3}. Since intK clK(A) ∩ clK intK(A) = {x, x+1, x+2, x+3} ⊆ A. Then by Definition 2.1(ii) A is b-closed. intK clK(A) ∪ clK intK(A) = {x-1, x, x+1, x+2, x+3} ⊆ A implies that A is b-open. Thus A is b-clopen. Also A ⊆ clK intK(A) = {x-1, x, x+1, x+2, x+3} then by Definition 2.1(iii), A is semi-open and intK clK(A) = {x, x+1, x+2} ⊆ A implies A is semi-closed. Also by Definition 2.1(v), A ⊆ clK intK clK(A) = {x-1, x, x+1, x+2, x+3} implies A is semi-pre-open and intK clK intK(A) = {x, x+1, x+2} ⊆ A implies A is semi-pre-closed. Since intK clK(A) ∪ clK intK(A) = {x-1, x, x+1, x+2, x+3} ⊆ A. Then by Definition 2.1(xi), A is b"-open and intK clK intK(A) ∪ clK intK clK(A) ⊆ A implies that A is b"-closed. Also by Definition 2.1(ix), intK clK(A) = {x, x+1, x+2} ⊆ clK intK(A) = {x-1, x, x+1, x+2, x+3} implies A is a q-set. Since intK(A) = intK clK(A), implies A is a t-set and clK(A) = clK intK(A) implies A is a t'-set.

Suppose x is even, intK(A) = {x+1, x+2, x+3} and clK intK(A) = {x, x+1, x+2, x+3, x+4}. 

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int_kcl(cl_k(A) = \{x+1, x+2, x+3\} and cl_k(A) = \{x, x+1, x+2, x+3, x+4\}. int_kcl(cl_k(A) = \{x+1, x+2, x+3\} and cl_k(\cap_k cl_k(A) = \{x+1, x+2, x+3\} \subset A, by Definition 2.1(vi), A is b-closed and int_k cl_k(A) \cup cl_k int_k(A) = \{x, x+1, x+2, x+3, x+4\}. Since int_k cl_k(A) \cap cl_k int_k(A) = \{x+1, x+2, x+3\} \subset A, by Definition 2.1(iii), A \subset cl_k int_k(A) = \{x, x+1, x+2, x+3\} \subset A implies A is b-open. Thus A is b-clopen. Now by Definition 2.1 (iii), A \subset cl_k int_k(A) = \{x, x+1, x+2, x+3, x+4\} implies A is semi-open and int_k cl_k(A) = \{x+1, x+2, x+3\} \subset A implies A is semi-closed. Since A \subset cl_k int_k cl_k(A) = \{x, x+1, x+2, x+3, x+4\} by Definition 2.1(v), A is semi-pre-open and int_k cl_k(A) = \{x+1, x+2, x+3\} \subset A implies A is semi-pre-closed. Also by Definition 2.1(xi), int_k cl_k(A) \cup cl_k int_k cl_k(A) = \{x, x+1, x+2, x+3, x+4\} \supset A implies A is b*-open and int_k cl_k(A) \cap cl_k int_k cl_k(A) \subset A implies A is b*-closed. Since int_k cl_k(A) = \{x+1, x+2, x+3\} by Definition 2.1(ii), A is a q-set. Now int_k cl_k(A) = int_k cl_k cl_k(A) implies A is a t-set and cl_k(A) = cl_k int_k(A) implies A is a t*-set. Therefore for every integer x, A is b-clopen, semi closed, semi open, semi-pre-open, semi-pre-closed, b*-open, b*-closed, a q-set, a t-set and a t*-set.

Proposition 3.12
In (Z, K), \{x, x+1, x+2, x+3\} is neither open nor closed.

Proof
Let A = \{x, x+1, x+2, x+3\}. Suppose x is odd. Then \{x+3\} is even, by Lemma 2.2(i), A is not open. Also by Lemma 2.2(ii), A is not closed. The proof for x is even is analog. Thus \{x, x+1, x+2, x+3\} is neither open nor closed.

Proposition 3.13
Suppose x is an odd integer. Then for every positive integer n \geq 2, the set A = \{x, x+1, x+2, \ldots, x+n-1, x+n\} is both b-closed and b-clopen in (Z, K).

Proof
Suppose n is even. Since x is odd, x+n is odd. Then int_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} and cl_k(A) = \{x-1, x, x+1, x+2, \ldots, x+n-1, x+n\}. Then int_k cl_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} and cl_k cl_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\}. Therefore int_k cl_k(A) \cap cl_k int_k(A) = A. Then by Definition 2.1(x), A is b-closed. Similarly int_k cl_k(A) \cup cl_k int_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} \supset A, then by Definition 2.1(vi), A is b-open. Since every b*-closed set is b-closed, A is b-closed.

Suppose n is odd. Since x is odd, x+n is even. Then int_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} and cl_k(A) = \{x-1, x, x+1, x+2, \ldots, x+n-1, x+n\}. Then int_k cl_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} and cl_k cl_k(A) = \{x-1, x, x+1, x+2, \ldots, x+n-1, x+n\}. Then int_k cl_k(A) \cap cl_k int_k(A) = A. Therefore by Definition 2.1(x), A is b*-closed. Similarly int_k cl_k(A) \cup cl_k int_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} \supset A, then by Definition 2.1(vi), A is b-open. Since every b*-closed set is b-closed, A is b-closed and b-clopen. Thus A is both b-closed and b-clopen for every positive integer n \geq 2.

Proposition 3.14
Suppose x is even. Then for every positive integer n, then the set A = \{x, x+1, x+2, \ldots, x+n-1, x+n\} is b-clopen in (Z, K).

Proof
Suppose n is even. Since x is even, x+n is odd. Then int_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} and cl_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\}. Then int_k cl_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} and cl_k cl_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\}. Then int_k cl_k(A) \cap cl_k int_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} \supset A. Then by Definition 2.1(vii), A is b-closed. Similarly int_k cl_k(A) \cup cl_k int_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} \supset A, then by Definition 2.1(vi), A is b-open. Thus A is b-clopen.

Suppose n is odd. Since x is even, x+n is even. Then int_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} and cl_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\}. Then int_k cl_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} and cl_k cl_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\}. Then int_k cl_k(A) \cap cl_k int_k(A) = \{x, x+1, x+2, \ldots, x+n-1, x+n\} \supset A. Then by Definition 2.1(vi), A is b-open. Thus A is b-clopen.
Definition 2.1(vi). If is b-closed. Similarly \( \text{int}_k \mathcal{A} \cup \text{cl}_k \mathcal{A} = \{ x, x+1, x+2, \ldots, x+n-1, x+n \} \supseteq A \). Then by Definition 2.1(vi), A is open. Thus A is b-clopen.

**Proposition 3.15**

In \((Z^2, K)\) the set \( \{(x, y)\} \) is regular open, pre-open, \( \alpha \)-open, semi-open, \( \beta \)-open, b-clopen, \( b^* \)-closed, semi-closed, \( \beta \)-closed, q-set, t-set, \( t^* \)-set if \( x \) and \( y \) are odd integers.

**Proof**

Let \( A = \{(x, y)\} \). Suppose \( x \) and \( y \) are odd. \( \text{int}_k (A) = \text{int}_k \{x\} \times \text{int}_k \{y\} = \{x\} \times \{y\} = (x, y) \). \( \text{cl}_k \{x\} \times \text{cl}_k \{y\} = \{x-1, x, x+1\} \times \{y-1, y, y+1\} = \{(x-1, y), (x, y), (x+1, y), (x-1, y+1), (x, y+1), (x+1, y+1)\} \supseteq (x, y) \).

\( A \subseteq \text{cl}_k \text{int}_k (A) \) implies by Definition 1.1(iii), A is semi-open. \( \text{cl}_k (A) = \text{cl}_k \{x\} \times \text{cl}_k \{y\} = \{x-1, x, x+1\} \times \{y-1, y, y+1\} = \{(x-1, y-1), (x-1, y), (x-1, y+1), (x, y), (x, y+1), (x+1, y), (x+1, y+1)\} \supseteq (x, y) \).

\( A \subseteq \text{cl}_k \text{int}_k (A) \) implies A is regular open by Definition 2.1(i). Also \( \text{int}_k \text{cl}_k (A) \subseteq A \) implies A is semi-closed and \( A \subseteq \text{int}_k \text{cl}_k (A) \) implies A is pre-open. \( \text{int}_k \text{cl}_k \text{int}_k (A) = \text{int}_k \text{cl}_k (A) \) implies A is \( \alpha \)-open by Definition 2.1(ii). Also \( \text{int}_k \text{cl}_k \text{int}_k (A) \subseteq A \) implies A is \( \beta \)-closed by Definition 2.1(v).

\( \text{cl}_k \text{int}_k \text{cl}_k (A) = \text{cl}_k \text{int}_k \text{cl}_k \{x\} \times \text{cl}_k \text{int}_k \{y\} = \{x-1, x, x+1\} \times \{y-1, y, y+1\} = \{(x-1, y-1), (x-1, y), (x-1, y+1), (x, y-1), (x, y), (x, y+1), (x+1, y-1), (x+1, y), (x+1, y+1)\} \supseteq (x, y) \).

\( A \subseteq \text{cl}_k \text{int}_k \text{cl}_k (A) \) implies A is semi-pre-open or \( \beta \)-closed. \( \text{int}_k \text{cl}_k \text{int}_k (A) \cap \text{cl}_k \text{int}_k (A) = \text{int}_k \text{cl}_k (A) \) if \( \text{int}_k (A) \subseteq \text{cl}_k \text{int}_k (A) \) and \( \text{cl}_k \text{int}_k (A) \cap \text{cl}_k (A) \) by Definition 2.1(x), A is b-closed. \( \text{int}_k \text{cl}_k (A) \cup \text{cl}_k \text{int}_k (A) = \text{cl}_k \text{int}_k (A) \times \text{cl}_k \text{int}_k (A) = \phi \times \phi = \phi \subseteq A \) implies A is pre-closed. \( \text{cl}_k (A) = \text{cl}_k \{x\} \times \text{cl}_k \{y\} = \phi \times \phi = \phi \subseteq A \) implies A is semi-closed. \( \text{int}_k \text{cl}_k \text{int}_k (A) = \text{int}_k \text{cl}_k \text{int}_k \{x\} \times \text{int}_k \text{cl}_k \text{int}_k \{y\} = \{x-1, x, x+1\} \times \{y-1, y, y+1\} = \{(x-1, y-1), (x-1, y), (x-1, y+1), (x, y-1), (x, y), (x, y+1), (x+1, y-1), (x+1, y), (x+1, y+1)\} \supseteq (x, y) \).

\( A \subseteq \text{int}_k \text{cl}_k \text{int}_k (A) \) implies A is \( \alpha \)-open by Definition 2.1(ii). \( \text{cl}_k \text{int}_k \text{cl}_k (A) = \text{cl}_k \text{int}_k \text{cl}_k \{x\} \times \text{cl}_k \text{int}_k \text{cl}_k \{y\} = \phi \times \phi = \phi \subseteq A \) implies A is \( \beta \)-closed by Definition 2.1(iv). \( \text{int}_k \text{cl}_k (A) \cap \text{cl}_k \text{int}_k \text{int}_k (A) = \text{int}_k \text{cl}_k \text{cl}_k \{x\} \times \text{cl}_k \text{int}_k \text{int}_k \{y\} = \phi \times \phi = \phi \subseteq A \) implies A is b-closed by Definition 2.1(vi). \( \text{int}_k \text{cl}_k \text{cl}_k (A) \cup \text{cl}_k \text{int}_k \text{int}_k (A) = \text{cl}_k \text{int}_k \text{int}_k \text{cl}_k \{x\} \times \text{cl}_k \text{int}_k \text{int}_k \text{cl}_k \{y\} \supseteq \text{cl}_k \text{int}_k \text{int}_k \text{cl}_k \{y\} = \phi \times \phi = \phi \subseteq A \) implies A is *b-closed. \( \text{cl}_k \text{cl}_k \{x\} \subseteq \text{cl}_k \text{cl}_k \{A\} \) implies by Definition 1.1(ix), A is a q-set and \( \text{cl}_k \text{cl}_k \{A\} \subseteq \text{cl}_k \text{cl}_k \{A\} \) implies by Definition 1.1(viii), A is a p-set. \( \text{cl}_k \{A\} = \text{cl}_k \text{int}_k \{A\} \) implies A is t-set and \( \text{int}_k \{A\} = \text{int}_k \text{cl}_k \{A\} \) implies A is a t-set.

Therefore A = (x, y) is semi-closed, pre-closed, \( \alpha \)-open, \( \alpha \)-closed, *b-closed, b-closed, a p-set, a q-set, a t-set and a t*-set if x and y are even integers. The proofs of (ii) and (iii) follow similarly.
CONCLUSION

In this paper, some nearly open sets, nearly closed sets including $b^\#$-closed sets were characterized in digital topology.

REFERENCES