Total Monophonic Eccentric Domination Number of Corona Product of Some Standard Graphs

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Abstract
For any two vertices x and y in a non-trivial connected graph G, the monophonic distance $d_m(x, y)$ is the length of a longest monophonic path joining the vertices x and y in G. The monophonic eccentricity of a vertex x is defined as $e_m(x) = \max \{d_m(x, y) : y \in V(G)\}$. A vertex y in G is a monophonic eccentric vertex of a vertex x in G if $e_m(x) = d_m(x, y)$. A set $S \subseteq V(G)$ is a total monophonic eccentric dominating set if every vertex in G has a monophonic eccentric vertex in S. The total monophonic eccentric domination number $\gamma_{tme}(G)$ is the cardinality of a minimum total monophonic eccentric dominating set of G. In this paper, we have identified the bounds of the total monophonic eccentric domination number of corona product of two graphs. Also, we have determined exact values of the total monophonic eccentric domination number of corona product of some standard graphs.

Keywords: monophonic path, monophonic distance, monophonic eccentric vertex, total monophonic eccentric dominating set, total monophonic eccentric domination number, corona.

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1. Introduction

Let $G = (V, E)$ be a finite undirected connected graph with $|V| = p$ and $|E| = q$. We refer [1, 5] for basic graph theoretic concepts and notations. The distance between any two vertices x and y is the length of a shortest path (geodesic) joining the vertices x and y, and it is denoted by $d(x, y)$. If $d(x, y) = 1$, then x is a neighbor of y and vice versa. A subset $S$ of the vertex set $V$ is called a dominating set if every vertex in $V - S$ has a neighbor in $S$. The domination number of $G$ is defined as $\gamma(G) = \min \{|S| : S$ is a dominating set of $G\}$. The idea of domination was introduced in [1] and further studied in [7]. Recently there are some new parameters introduced based on domination, and a textbook [6] on domination was published in 1998. Also, total domination in graphs was introduced in [4].

The detour distance between any two vertices x and y is the length of a longest path (detour) joining the vertices x and y, and it is denoted by $D(x, y)$. A vertex y is called a detour neighbor of a vertex x if $D(x, y) \leq D(x, z)$ for any $z \in V - \{x, y\}$. A subset $S$ of the vertex set $V$ is called a detour dominating set if every vertex in $V - S$ has a detour neighbor in $S$ and the detour domination number is defined as $\gamma_D(G) = \min \{|S| : S$ is a
The concept of detour domination was introduced and studied in [3].

A chordless path is also called as a monophonic path. The monophonic distance between any two vertices \(x\) and \(y\) is the length of a longest monophonic path joining the vertices \(x\) and \(y\), and it is denoted by \(d_m(x,y)\). For any vertex \(x\) in \(G\), the monophonic eccentricity of a vertex \(x\) is defined as \(e_m(x) = \max \{d_m(x,y) : y \in V\}\). A vertex \(y\) in \(G\) is a monophonic eccentric vertex of a vertex \(x\) in \(G\) if \(e_m(x) = d_m(x,y)\). The monophonic radius \(\text{rad}_m(G)\) is the minimum monophonic eccentricity among the vertices of \(G\) and the monophonic diameter \(\text{diam}_m(G)\) is the maximum monophonic eccentricity among the vertices of \(G\). In [8, 9], Santhakumaran and Titus initiated the study of monophonic distance and further related results.

The monophonic eccentric dominating set and the monophonic eccentric domination number of a graph were introduced and studied in [11, 13]. The total monophonic eccentric domination number was introduced and studied in [12]. The parameters monophonic eccentric domination number and total monophonic eccentric domination number have many useful applications in channel assignment problems in radio technologies. Further, these concepts have huge amount of application in molecular problems in theoretical chemistry.

The following definitions and theorems will be used in the sequel.

**Definition 1.1** [11] A set \(S \subseteq V\) in a graph \(G\) is a monophonic eccentric dominating set if every vertex in \(V - S\) has a monophonic eccentric vertex in \(S\). The monophonic eccentric domination number \(\gamma_{me}(G)\) is the cardinality of a minimum monophonic eccentric dominating set of \(G\).

**Definition 1.2** [12] A set \(S \subseteq V\) in a graph \(G\) is a total monophonic eccentric dominating set if every vertex in \(G\) has a monophonic eccentric vertex in \(S\). The total monophonic eccentric domination number \(\gamma_{tme}(G)\) is the cardinality of a minimum total monophonic eccentric dominating set of \(G\).

For the graph \(G\) given in Figure 1. A graph \(G\) with \(\gamma_{me}(G) = 2\) and \(\gamma_{tme}(G) = 4\), it can be easily verify that \(S = \{v_2,v_3\}\) is a minimum monophonic eccentric dominating set so that \(\gamma_{me}(G) = 2\). But \(S\) is not a total monophonic eccentric dominating set of \(G\). Also, no two or three element subset of \(V(G)\) is a total monophonic eccentric dominating set. It is clear that \(S_1 = \{v_2,v_3,v_5,v_6\}\) is a total monophonic eccentric dominating set of \(G\) so that \(\gamma_{tme}(G) = 4\).

![Figure 1](image-url)
Theorem 1.3 [12] If $G = H + K_p$, where $H$ is any connected graph, then $\gamma_{tme}(G) = \gamma_{tme}(H)$.

Theorem 1.4 [12] Let $G$ be a cycle of order $p$ and let $p \equiv l \pmod{8}$. Then

$$\gamma_{tme}(G) = \begin{cases} 
\frac{p+1}{2} & \text{if } l \text{ is odd} \\
\frac{p+l}{2} & \text{if } l = 0, 2 \text{ or } 4 \\
\frac{p+2}{2} & \text{if } l = 6.
\end{cases}$$

Theorem 1.5 [12] Let $G$ be a wheel of order $p$ and let $p \equiv l \pmod{8}$. Then

$$\gamma_{tme}(G) = \begin{cases} 
\frac{p}{2} & \text{if } l \text{ is even} \\
\frac{p-2+l}{2} & \text{if } l = 1, 3 \text{ or } 5 \\
\frac{p+1}{2} & \text{if } l = 7.
\end{cases}$$

2. Total Monophonic Eccentric Domination Number

The corona of two graphs $G_1$ and $G_2$ is the graph $G = G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the $i^{th}$ vertex of $G_1$ is adjacent to every vertex in the $i^{th}$ copy of $G_2$. The distance related properties of corona was studied in [10] and the domination parameters of corona was studied in [2].

Theorem 2.1 Let $G$ be a connected graph of order $m$ and let $H$ be any graph of order $n$. Then $2 \leq \gamma_{tme}(G \circ H) \leq m \gamma_{tme}(H)$.

Proof. Since every total monophonic eccentric dominating set of a graph $G \circ H$ contains at least two monophonic eccentric vertices, $\gamma_{tme}(G \circ H) \geq 2$. Let $H_{i,n}$ be the $i^{th}$ copy of $H$ $(1 \leq i \leq m)$. Let $S_i$ be a minimum total monophonic eccentric dominating set of $H_{i,n}$. In $G \circ H$, it is clear that every vertex in $H_{i,n}$ is monophonic eccentric dominated by a vertex in $S_i$ and every vertex in $G$ is monophonic eccentric dominated by a vertex in $S = \bigcup_{i=1}^{m} S_i$. Hence $S$ is a total monophonic eccentric dominating set of $G \circ H$ and so $\gamma_{tme}(G \circ H) \leq m \gamma_{tme}(H)$.

Remark 2.2 The bounds in Theorem 2.1 are sharp. For the graph $G=K_1 \circ K_n$, $\gamma_{tme}(G) = 2$ and for the graph $G = C_m \circ C_n$ where $n \geq m+3$, $\gamma_{tme}(G) = m \gamma_{tme}(C_n)$.

Theorem 2.3 Let $G$ be a connected graph of order $m$ and let $H$ be any graph of order $n$. If $diam_m(H) \leq diam_m(G) + 2$, then $\gamma_{tme}(G \circ H) \leq m$.

Proof. Let $u_1, u_2, \ldots, u_m$ be the vertices of $G$ and let $H_{i,n}$ be the $i^{th}$ copy of $H$ with the vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,n}$ $(1 \leq i \leq m)$. Since $diam_m(H) \leq diam_m(G) + 2$, every vertex in $H_{i,n}$ $(1 \leq i \leq m)$ is monophonic eccentric dominated by at least one vertex in $H_{j,n}$ $(1 \leq j \leq m)$.
and $i \neq j$) and every vertex in $G$ is monophonic eccentric dominated by at least one vertex in $H_{i,n}$ ($1 \leq i \leq m$). Let $S = \{v_{1,j}, v_{2,j}, \ldots, v_{m,j}\}$ ($1 \leq j \leq n$). Hence every element in $V(G \circ H)$ is monophonic eccentric dominated by a vertex in $S$. Thus $S$ is a total monophonic eccentric dominating set of $G$ and so $\gamma_{tme}(G \circ H) \leq m$. 

**Remark 2.4** The bound in Theorem 2.3 is sharp. For the graph $G = C_4 \circ P_s$, where $1 \leq s \leq 5$, $\gamma_{tme}(G) = 4$.

**Theorem 2.5** If $G = P_r \circ P_s$ ($r \geq 3$), then

$$\gamma_{tme}(G) = \begin{cases} 2 & \text{if } 1 \leq s \leq \left\lfloor \frac{r+6}{2} \right\rfloor \\ 2(2s-r-5) & \text{if } s = \left\lfloor \frac{r+8}{2} \right\rfloor, \left\lfloor \frac{r+10}{2} \right\rfloor, \ldots, r+2 \\ 2r & \text{if } s > r+2. \end{cases}$$

**Proof.** Let $u_1, u_2, \ldots, u_t$ ($r \geq 3$) and $v_{1,i}, v_{2,j}, \ldots, v_{s,k}$ be the vertices of $P_r$ and $i^{th}$ copy of $P_s$ ($1 \leq i \leq r$), respectively. We prove this theorem by considering three cases based on $r$ and $s$.

**Case 1.** $1 \leq s \leq \left\lfloor \frac{r+6}{2} \right\rfloor$.

Let $S = \{v_{1,j}, v_{r,j}\}$ ($1 \leq j \leq s$). If $r$ is even, then the vertices $u_i$ ($1 \leq i \leq \frac{r}{2}$) and $v_{i,j}$ ($1 \leq i \leq \frac{r}{2}$) are monophonic eccentric dominated by the vertex $v_{r,j}$ and the vertices $u_i$ ($\frac{r}{2} \leq i \leq r$) and $v_{i,j}$ ($\frac{r}{2} \leq i \leq r$, $1 \leq j \leq s$) are monophonic eccentric dominated by the vertex $v_{i,j}$. If $r$ is odd, then the vertices $u_i$ ($1 \leq i \leq \frac{r-1}{2}$) and $v_{i,j}$ ($1 \leq i \leq \frac{r-1}{2}$, $1 \leq j \leq s$) are monophonic eccentric dominated by the vertex $v_{i,j}$, the vertices $u_i$ ($\frac{r+3}{2} \leq i \leq r$) and $v_{i,j}$ ($\frac{r+3}{2} \leq i \leq r$, $1 \leq j \leq s$) are monophonic eccentric dominated by the vertex $v_{i,j}$, and the vertices $u_{\frac{r+1}{2}}$ and $v_{\frac{r+1}{2}}$ ($1 \leq j \leq s$) are monophonic eccentric dominated by both the vertices $v_{1,j}$ and $v_{r,j}$. Hence $S$ is a minimum total monophonic eccentric dominating set of $G$ and so $\gamma_{tme}(G) = 2$.

**Case 2.** $s = \left\lfloor \frac{r+8}{2} \right\rfloor, \left\lfloor \frac{r+10}{2} \right\rfloor, \ldots, r+2$.

**Subcase (i)** $r$ is even.

Let $m = s - \frac{r+6}{2}$. It is clear that the vertices $u_i$ ($1 \leq i \leq \frac{r}{2}$) and $v_{i,j}$ ($1 \leq i \leq \frac{r}{2} - m$, $1 \leq j \leq s$) are monophonic eccentric dominated by the vertex $v_{r,j}$, the vertices $u_i$ ($\frac{r}{2} \leq i \leq r$) and $v_{i,j}$ ($\frac{r}{2} \leq i \leq r$, $1 \leq j \leq s$) are monophonic eccentric dominated by the vertex...
the vertices \( v_{r+2_{-m,l}} \) (2 \( \leq l \leq s-1 \)) are monophonic eccentric dominated by the vertex

\( v_{r+2_{-m,l}} \), the vertices \( v_{r+2_{+m,l}} \) (2 \( \leq l \leq s-1 \)) are monophonic eccentric dominated by the vertex \( v_{r+2_{-m,l}} \),

the vertex \( v_{r+2_{-m,l}} \) is monophonic eccentric dominated by the vertex \( v_{r+2_{-m,l}} \), the vertex \( v_{r+2_{+m,l}} \) is monophonic eccentric dominated by the vertex \( v_{r+2_{-m,l}} \) and the vertex \( v_{r+2_{+m,l}} \) is monophonic eccentric dominated by the vertex \( v_{r+2_{+m,l}} \).

Also, the vertices \( v_{k,l} (\frac{r+4}{2} - m \leq 1 \leq 2 \leq m+k - \frac{r}{2} ) \) are monophonic eccentric dominated by the vertex \( v_{k,l} \), the vertices \( v_{k,l} (\frac{r+4}{2} - m \leq \frac{r+2}{2} \leq k \leq \frac{r-2}{2} + m \), \( 1 \leq l \leq s-k-2 \) are monophonic eccentric dominated by the vertex \( v_{k,l} \), the vertices \( v_{k,l} (\frac{r+2}{2} \leq k \leq \frac{r-2}{2} + m \), \( s-k-1 \leq l \leq k+2 \) are monophonic eccentric dominated by the vertex \( v_{k,l} \), and the vertices \( v_{k,l} (\frac{r+2}{2} \leq k \leq \frac{r-2}{2} + m \), \( k+3 \leq l \leq s \) are monophonic eccentric dominated by the vertex \( v_{k,l} \).

Hence \( S=\{ v_{l,j}, v_{r+2_{-m,l}}, v_{r+2_{-m,s}}, v_{r+2_{+m,l}}, v_{r+2_{+m,s}} \} \)

\( \cup \{ v_{r+4_{-m,l}}, v_{r+4_{-m,s}}, ... , v_{r+4_{-s}}, v_{r+4_{-s}}, ... , v_{r+4_{-s}} \} \) is a minimum total monophonic eccentric dominating set of \( G \) and so

\[ \gamma_{te}(G) = 6+4(m-1) \]

\[ = 6+4(s \cdot \frac{r+6}{2} - 1) \]

\[ = 2 (2s-r-5). \]

**Subcase (ii)** \( r \) is odd.

Let \( m = s - \frac{r+7}{2} \). It is clear that the vertices \( u_i (1 \leq i \leq \frac{r-1}{2} ) \) and \( v_{l,j} (1 \leq i \leq \frac{r-1}{2} - m, \)

\( 1 \leq j \leq s \) are monophonic eccentric dominated by the vertex \( v_{l,j} \), the vertices \( u_i (\frac{r+3}{2} \leq i \leq r) \) and \( v_{l,j} (\frac{r+3}{2} + m \leq i \leq r, 1 \leq j \leq s) \) are monophonic eccentric dominated by the vertex \( v_{l,j} \), the vertices \( v_{r+1_{-m,l}} (2 \leq l \leq s-1) \) are monophonic eccentric dominated by the vertex \( v_{l,j} \), the vertices \( v_{r+1_{-m,l}} (2 \leq l \leq s-1) \) are monophonic eccentric dominated by the vertex \( v_{l,j} \), the vertices...
by the vertex \( v_{l,j} \), the vertex \( v_{r+1,2m,s} \) is monophonic eccentric dominated by the vertex

\[
v_{r+1,2m-l,j} \quad \text{and the vertices} \quad v_{r+1,2m-s,j} \quad \text{are monophonic eccentric dominated by the vertex} \quad v_{r+1,2m+s,j} \quad \text{and the vertex} \quad v_{r+1,2m+1,j}.
\]

Also, the vertices \( u_{r+1,2} \) and \( v_{r+1,2j} \) are monophonic eccentric dominated by both the vertices \( v_{l,j} \) and \( v_{r,j} \), the vertices \( v_{r+1,2m,j} \) (1 \( \leq j \leq m+1 \)) are monophonic eccentric dominated by the vertex \( v_{r+1,2} \) and the vertices \( v_{r+1,2j} \) (\( s-m \leq j \leq s \)) are monophonic eccentric dominated by the vertex \( v_{r+1,2} \), the vertices \( v_{k,l} \) (\( r+3 \leq m \leq k \leq r-1 \), 1 \( \leq l \leq m+k-r-1 \)) are monophonic eccentric dominated by the vertex \( v_{k,s} \), the vertices \( v_{k,l} \) (\( r+3 \leq m \leq k \leq r-1 \), \( m+k-r-3 \leq l \leq r-k+3 \)) are monophonic eccentric dominated by the vertex \( v_{r,j} \), the vertices \( v_{k,l} \) (\( r+3 \leq m \leq k \leq r-1 \), \( r-k+4 \leq l \leq s \)) are monophonic eccentric dominated by the vertex \( v_{k,l} \), the vertices \( v_{k,l} \) (\( r+3 \leq m \leq k \leq r-1 \), 1 \( \leq l \leq s-k-2 \)) are monophonic eccentric dominated by the vertex \( v_{k,s} \), the vertices \( v_{k,l} \) (\( r+3 \leq m \leq k \leq r-1 \), \( s-k-1 \leq l \leq k+2 \)) are monophonic eccentric dominated by the vertex \( v_{l,j} \) and the vertices \( v_{r,j} \) (\( k \leq \frac{r-1}{2} \), \( k+3 \leq l \leq s \)) are monophonic eccentric dominated by the vertex \( v_{k,l} \). Hence

\[
S = \{ v_{l,j}, v_{r+1,2m-s}, v_{r+1,2m-t} \} \cup \{ v_{r+1,2m+1-s}, v_{r+1,2m+1-t} \} \cup \{ v_{r+1,2m-s}, v_{r+1,2m-t} \}
\]

is a minimum total monophonic eccentric dominating set of \( G \) and so

\[
\gamma_{tme}(G) = 8+4(m-1)
\]

\[
= 8+4(s-\frac{r+7}{2}-1)
\]

\[
= 2(2s-r-5).
\]

**Case 3.** \( s > r+2 \).

Let \( m = \left\lfloor \frac{s-(r+2)}{2} \right\rfloor \). If \( r \) is even, then the vertices \( u_i \) (1 \( \leq i \leq \frac{r}{2} \)) are monophonic eccentric dominated by the vertex \( v_{r,j} \) and the vertices \( u_i \) (\( \frac{r+2}{2} \leq i \leq r \)) are.
monophonic eccentric dominated by the vertex $v_{i,j}$. In $1 \leq k \leq \frac{r}{2}$, if $m + k \leq \frac{r + 2}{2}$, then the vertices $v_{k,l}$ ($1 \leq l \leq s+k-r-3$) are monophonic eccentric dominated by the vertex $v_{k,s}$, the vertices $v_{k,l}$ ($s+k-r-2 \leq l \leq r+3-k$) are monophonic eccentric dominated by the vertex $v_{r,j}$, and the vertices $v_{k,l}$ ($r+4-k \leq l \leq s$) are monophonic eccentric dominated by the vertex $v_{k,l}$. In $1 \leq k \leq \frac{r}{2}$, if $m + k > \frac{r + 2}{2}$, then the vertices $v_{k,l}$ ($1 \leq l \leq \left\lfloor \frac{s+1}{2} \right\rfloor$) are monophonic eccentric dominated by the vertex $v_{k,s}$ and the vertices $v_{k,l}$ ($\left\lfloor \frac{s+3}{2} \right\rfloor \leq l \leq s$) are monophonic eccentric dominated by the vertex $v_{k,l}$.

If $r$ is odd, then the vertices $u_{i}$ ($1 \leq i \leq \frac{r-1}{2}$) are monophonic eccentric dominated by the vertex $v_{r,j}$, the vertex $u_{\frac{r+1}{2}}$ is monophonic eccentric dominated by both the vertices $v_{r,j}$ and $v_{r,j}$, and the vertices $u_{i}$ ($\frac{r+3}{2} \leq i \leq r$) are monophonic eccentric dominated by the vertex $v_{r,j}$. In $1 \leq k \leq \frac{r+1}{2}$, if $m + k \leq \frac{r+1}{2}$, then the vertices $v_{k,j}$ ($1 \leq l \leq s+k-r-3$) are monophonic eccentric dominated by the vertex $v_{k,s}$, the vertices $v_{k,j}$ ($1 \leq k \leq \frac{r-1}{2}$, $s+k-r-2 \leq l \leq r+3-k$) are monophonic eccentric dominated by the vertex $v_{r,j}$, the vertices $v_{r,j}$ ($s+k-r-2 \leq l \leq r+3-k$) are monophonic eccentric dominated by both the vertices $v_{r,j}$ and $v_{r,j}$, and the vertices $v_{k,j}$ ($r+4-k \leq l \leq s$) are monophonic eccentric dominated by the vertex $v_{k,j}$. In $1 \leq k \leq \frac{r+1}{2}$, if $m + k > \frac{r+1}{2}$, then the vertices $v_{k,l}$ ($1 \leq l \leq \left\lfloor \frac{s+1}{2} \right\rfloor$) are monophonic eccentric dominated by the vertex $v_{k,s}$ and the vertices $v_{k,l}$ ($\left\lfloor \frac{s+3}{2} \right\rfloor \leq l \leq s$) are monophonic eccentric dominated by the vertex $v_{k,l}$. In $\frac{r+3}{2} \leq k \leq r$,
if \( k-m \geq \frac{r+1}{2} \), then the vertices \( v_{k,l} (1 \leq l \leq s-k-2) \) are monophonic eccentric dominated by the vertex \( v_{k,s} \), the vertices \( v_{k,j} (s-k-1 \leq l \leq k+2) \) are monophonic eccentric dominated by the vertex \( v_{l,j} \), and the vertices \( v_{k,j} (k+3 \leq l \leq s) \) are monophonic eccentric dominated by the vertex \( v_{k,s} \). In \( \frac{r+3}{2} \leq k \leq r \), if \( k-m < \frac{r+1}{2} \), then the vertices \( v_{k,l} (1 \leq l \leq \left\lfloor \frac{s+1}{2} \right\rfloor ) \) are monophonic eccentric dominated by the vertex \( v_{k,s} \) and the vertices \( v_{k,l} (\left\lfloor \frac{s+3}{2} \right\rfloor \leq l \leq s) \) are monophonic eccentric dominated by the vertex \( v_{k,l} \). Hence \( S=\{v_{1,1}, v_{2,1}, \ldots, v_{l,1}\} \cup \{v_{1,s}, v_{2,s}, \ldots, v_{r,s}\} \) is a minimum total monophonic eccentric dominating set of \( G \) and so \( \gamma_{\text{tme}}(G) = r + r = 2r \).

**Proof.**

Let \( G \) be the corona product of \( P_r \) and \( C_s \). Let \( u_{i,j}, u_{2}, \ldots, u_{r} \) be the vertices of \( P_r \) and let \( C_{s,1}: v_{1,1}, v_{1,2}, \ldots, v_{1,s}, v_{i,j} \) be the \( i \)th copy of \( C_s \) \((1 \leq i \leq r)\). We prove this theorem by considering three cases.

**Case 1.** \( 3 \leq s \leq \left\lfloor \frac{r+7}{2} \right\rfloor \).

If \( r \) is even, then the vertices \( u_i \) and \( v_{i,j} (1 \leq i \leq \frac{r}{2}, 1 \leq j \leq s) \) are monophonic eccentric dominated by the vertex \( v_{r,j} \) and the vertices \( u_i \) and \( v_{i,j} (\frac{r}{2} \leq i \leq r, 1 \leq j \leq s) \) are monophonic eccentric dominated by the vertex \( v_{l,j} \). Then \( S=\{v_{i,j}, v_{r,j}\} \) is a minimum total monophonic eccentric dominating set of \( G \) and so \( \gamma_{\text{tme}}(G)=2 \).

If \( r \) is odd, then the vertices \( u_i \) and \( v_{i,j} (1 \leq i \leq \frac{r-1}{2}, 1 \leq j \leq s) \) are monophonic eccentric dominated by the vertex \( v_{r,j} \), the vertices \( u_i \) and \( v_{i,j} (\frac{r+3}{2} \leq i \leq r, 1 \leq j \leq s) \) are monophonic eccentric dominated by the vertex \( v_{l,j} \), and the vertices \( u_{i+j} \) and \( v_{j} \) \((1 \leq j \leq s)\) are monophonic eccentric dominated by the vertex \( v_{l,j} \). Then \( S=\{v_{i,j}, v_{r,j}, u_{i+j}, v_{j}\} \) is a minimum total monophonic eccentric dominating set of \( G \) and so \( \gamma_{\text{tme}}(G)=2 \).

**Note 2.6**

(i) If \( G = P_1 \circ P_s \), then \( \gamma_{\text{tme}}(G) = 2 \).

(ii) If \( G = P_1 \circ P_s \), then \( \gamma_{\text{tme}}(G) = \begin{cases} 2 & \text{if } 1 \leq s \leq \left\lfloor \frac{r+6}{2} \right\rfloor \\ 2r & \text{if } s > r+2 \end{cases} \)

**Theorem 2.7** If \( G = P_r \circ C_s \) \((r \geq 3)\), then

\[
\gamma_{\text{tme}}(G) = \begin{cases} 2 & \text{if } 3 \leq s \leq \left\lfloor \frac{r+7}{2} \right\rfloor \\ (2s - r - 8)\gamma_{\text{tme}}(C_s) + 2 & \text{if } s = \left\lceil \frac{r+9}{2} \right\rceil, \left\lceil \frac{r+11}{2} \right\rceil, \ldots, r+3 \\ r\gamma_{\text{tme}}(C_s) & \text{if } s > r + 3. \end{cases}
\]
≤ s) are monophonic eccentric dominated by both the vertices \(v_{i,j}\) and \(v_{r,j}\). Hence, it is clear that \(S = \{v_{i,j}, v_{r,j}\}\) is a minimum total monophonic eccentric dominating set of \(G\) and so \(\gamma_{tme}(G) = 2\).

Case 2. \(s = \left\lfloor \frac{r + 9}{2} \right\rfloor, \left\lfloor \frac{r + 11}{2} \right\rfloor, \ldots, r + 3\).

Subcase (i) \(r\) is even.

Let \(m = s - \frac{r + 8}{2}\). It can be easily seen that the vertices \(u_i (1 \leq i \leq \frac{r}{2})\) and \(v_{i,j} (1 \leq i \leq \frac{r}{2}, 1 \leq j \leq s)\) are monophonic eccentric dominated by the vertex \(v_{r,j}\). Similarly, the vertices \(u_i \left(\frac{r + 2}{2} \leq i \leq r\right)\) and \(v_{i,j} \left(\frac{r + 2}{2} + m \leq i \leq r, 1 \leq j \leq s\right)\) are monophonic eccentric dominated by the vertex \(v_{i,j}\). Let \(S_k \left(\frac{r + 2}{2} - m \leq k \leq \frac{r}{2} + m\right)\) be a minimum total monophonic eccentric dominating set of \(C_{k,s}\). It is clear that, in \(G\), any vertex in \(C_{k,s}\) \((\frac{r + 2}{2} - m \leq k \leq \frac{r + m}{2})\) is monophonic eccentric dominated by a vertex in \(S_k\) and hence

\[
S = \bigcup_{i \geq r/2} S_i \bigcup \{v_{1,j}, v_{r,j}\} \text{ is a minimum total monophonic eccentric dominating set of } G.
\]

Thus \(\gamma_{tme}(G) = |S| = 2m \gamma_{tme}(C_s) + 2 = 2 \left\lfloor \left(s - \frac{r + 8}{2} \right) \right\rfloor \gamma_{tme}(C_s) + 2 = 2(r - s - 8 \gamma_{tme}(C_s) + 2)

Subcase (ii) \(r\) is odd.

Let \(m = s - \frac{r + 7}{2}\). It can be easily seen that the vertices \(u_i (1 \leq i \leq \frac{r - 1}{2})\) and \(v_{i,j} (1 \leq i \leq \frac{r - 1}{2}, 1 \leq j \leq s)\) are monophonic eccentric dominated by the vertex \(v_{r,j}\). Similarly, the vertices \(u_i \left(\frac{r + 3}{2} \leq i \leq r\right)\) and \(v_{i,j} \left(\frac{r + 1}{2} + m \leq i \leq r, 1 \leq j \leq s\right)\) are monophonic eccentric dominated by the vertex \(v_{i,j}\). Also, the vertex \(u_{r+1j}\) is monophonic eccentric dominated by both the vertices \(v_{i,j}\) and \(v_{r,j}\). Let \(S_k \left(\frac{r + 3}{2} - m \leq k \leq \frac{r - 1}{2} + m\right)\) be a minimum total monophonic eccentric dominating set of \(C_{k,s}\). It is clear that, in \(G\), any vertex in \(C_{k,s}\) \((\frac{r + 3}{2} - m \leq k \leq \frac{r - 1}{2} + m)\) is monophonic eccentric dominated by a vertex
in \( S_k \) and hence \( S = \bigcup_{i=2}^{r-1} S_i \cup \{ v_{1,j}, v_{r,j} \} \) is a minimum total monophonic eccentric dominating set of \( G \). Thus

\[
\gamma_{tme}(G) = 8 \\
= (2m-1) \gamma_{tme}(Cs) + 2 \\
= \left( 2 \left( s - \frac{r+7}{2} \right) - 1 \right) \gamma_{tme}(Cs) + 2 \\
= (2s-r-8) \gamma_{tme}(Cs) + 2.
\]

**Case 3.** \( s > r + 3 \).

If \( r \) is even, then the vertices \( u_i \) \( (1 \leq i \leq \frac{r}{2}) \) are monophonic eccentric dominated by a vertex \( v_{r,j} \) \( (1 \leq j \leq s) \) and the vertices \( u_i \) \( (\frac{r+2}{2} \leq i \leq r) \) are monophonic eccentric dominated by a vertex \( v_{l,j} \) \( (1 \leq j \leq s) \). If \( r \) is odd, then the vertices \( u_i \) \( (1 \leq i \leq \frac{r-1}{2}) \) are monophonic eccentric dominated by a vertex \( v_{r,j} \) \( (1 \leq j \leq s) \), the vertices \( u_i \) \( (\frac{r+3}{2} \leq i \leq r) \) are monophonic eccentric dominated by a vertex \( v_{l,j} \) \( (1 \leq j \leq s) \), and the vertex \( u_{r+1} \) is monophonic eccentric dominated by both the vertices \( v_{l,j} \) and \( v_{r,j} \) \( (1 \leq j \leq s) \).

Let \( S_k \) \( (1 \leq k \leq r) \) be a minimum total monophonic eccentric dominating set of \( C_{k,s} \). It is clear that, in \( G \), any vertex in \( C_{k,s} \) \( (1 \leq k \leq r) \) is monophonic eccentric dominated by a vertex in \( S_k \) and hence \( S = \bigcup_{i=2}^{r-1} S_i \) is a minimum total monophonic eccentric dominating set of \( G \). Thus

\[
\gamma_{tme}(G) = r \gamma_{tme}(Cs).
\]

The result of the above theorem contains \( \gamma_{tme}(Cs) \) and we can calculate \( \gamma_{tme}(Cs) \) using Theorem 1.2.

**Note 2.8** (i) If \( G = P_1 \circ Cs \), then \( G \) is a wheel. Hence the value of \( \gamma_{tme}(G) \) is calculated using Theorem 1.3.

(ii) If \( G = P_2 \circ Cs \), then \( \gamma_{tme}(G) = \begin{cases} 
2 & \text{if } 3 \leq s \leq \left\lfloor \frac{r+7}{2} \right\rfloor \\
r \gamma_{tme}(Cs) & \text{if } s > r+3
\end{cases} \)

**Theorem 2.9** If \( G = P_r \circ W_s \) \( (r \geq 3) \), then

\[
\gamma_{tme}(G) = \begin{cases} 
2 & \text{if } 4 \leq s \leq \left\lfloor \frac{r+9}{2} \right\rfloor \\
(2s-r-10)\gamma_{tme}(W_s) + 2 & \text{if } s = \left\lfloor \frac{r+11}{2} \right\rfloor, \left\lfloor \frac{r+13}{2} \right\rfloor, \ldots, r+4 \\
r \gamma_{tme}(W_s) & \text{if } s > r+4
\end{cases}
\]


**Proof.** Since \( W_s = K_1 + C_{s-1} \), by Theorem 1.1, we have \( \gamma_{tme}(W_s) = \gamma_{tme}(C_{s-1}) \). Then by an argument similar to Theorem 2.7, the required result can be got.

**Note 2.10** (i) If \( G = P_1 \circ W_s = K_1 + W_s \). By Theorem 1.1, we have \( \gamma_{tme}(G) = \gamma_{tme}(W_s) \).

(ii) If \( G = P_2 \circ W_s \), then \( \gamma_{tme}(G) = \begin{cases} 
2 & \text{if } 4 \leq s \leq \left\lfloor \frac{r+9}{2} \right\rfloor \\
r\gamma_{tme}(W_s) & \text{if } s > r+4.
\end{cases} \)

**Theorem 2.11** If \( G = P_r \circ K_s \) or \( G = P_r \circ K_{l,n} \), then \( \gamma_{tme}(G) = 2 \).

**Proof.** Let \( G = P_r \circ K_s \) or \( G = P_r \circ K_{l,n} \). Let \( u_1, u_2, \ldots, u_r \) be the vertices of \( P_r \). Let \( H = K_s \) or \( K_{l,n} \) and let \( H_i (1 \leq i \leq r) \) be the copies of \( H \). If \( r = 1 \), let \( S = \{x, y\} \) where \( x \) and \( y \) are any pair of monophonic antipodal vertices of \( G \). Then any vertex in \( G - x \) is monophonic eccentric dominated by the vertex \( x \) and the vertex \( x \) is monophonic eccentric dominated by the vertex \( y \). Hence \( S \) is a minimum total monophonic eccentric dominating set of \( G \) and so \( \gamma_{tme}(G) = 2 \). If \( r > 1 \), let \( S_i = \{u, v\} \), where \( u \in V(H_1) \) and \( v \in V(H_r) \). It is easily verified that the vertices of \( H_i \) and the vertices \( u_i \left( \left\lfloor \frac{r+3}{2} \right\rfloor \leq i \leq r \right) \) are monophonic eccentric dominated by the vertex \( u \), the vertices of \( H_j \) and the vertices \( u_j (1 \leq j \leq \left\lfloor \frac{r+1}{2} \right\rfloor) \) are monophonic eccentric dominated by the vertex \( v \). It is clear that \( S_i \) is a minimum total monophonic eccentric dominating set of \( G \) and so \( \gamma_{tme}(G) = 2 \).

**Theorem 2.12** If \( G = P_r \circ K_{m,n} (m, n \geq 2) \), then \( \gamma_{tme}(G) = \begin{cases} 
4 & \text{if } r = 1 \\
2 & \text{if } r > 1.
\end{cases} \)

**Proof.** Let \( G \) be the corona product of \( P_r \) and \( K_{m,n} \). Let \( u_1, u_2, \ldots, u_r \) be the vertices of \( P_r \) and let \( V_{i,1} = \{v_{i,1}, v_{i,2}, \ldots, v_{i,m}\} \) and \( V_{i,2} = \{w_{i,1}, w_{i,2}, \ldots, w_{i,n}\} \) be the partite sets of the \( i \)-th copy of \( K_{m,n} \) \( (1 \leq i \leq r) \). If \( r = 1 \), let \( S = \{v_{1,1}, v_{1,2}, w_{1,1}, w_{1,2} \} \) \( (2 \leq i \leq m, 2 \leq j \leq n) \). It can be easily seen that the vertices \( v_{i,1} \) \( (2 \leq i \leq m) \) and \( u_1 \) are monophonic eccentric dominated by the vertex \( v_{i,1} \), the vertex \( v_{i,1} \) is monophonic eccentric dominated by a vertex \( v_{i,1} \), the vertices \( w_{i,j} \) \( (2 \leq j \leq n) \) are monophonic eccentric dominated by the vertex \( w_{i,j} \) and the vertex \( w_{i,j} \) is monophonic eccentric dominated by a vertex \( w_{i,j} \). Hence \( S \) is a minimum total monophonic eccentric dominating set of \( G \) and so \( \gamma_{tme}(G) = 4 \). If \( r > 1 \), let \( S_i = \{v_{i,j}, v_{i,j} \} \) \( (1 \leq j \leq m) \). It is easily verified that the vertices \( u_i, v_{i,j} \) and \( w_{i,j} \left( \left\lfloor \frac{r+3}{2} \right\rfloor \leq i \leq r, 1 \leq j \leq m, 1 \leq k \leq n \right) \) are monophonic eccentric dominated by a vertex \( v_{i,j} \) and the vertices \( u_i, v_{i,j} \) and \( w_{i,k} \left( 1 \leq i \leq \left\lfloor \frac{r+1}{2} \right\rfloor, 1 \leq j \leq m, 1 \leq k \leq n \right) \) are monophonic eccentric dominated by a vertex \( v_{i,j} \) \( (1 \leq j \leq m) \). Hence \( S_i \) is a minimum total monophonic eccentric dominating set of \( G \) and so \( \gamma_{tme}(G) = 2 \).

**Theorem 2.13** If \( G = C_r \circ P_s \), then

\[
\gamma_{tme}(G) = \begin{cases} 
\frac{r+1}{2} & \text{if } 1 \leq s \leq r + 1 \text{ and } r \text{ is odd} \\
\frac{r}{2} & \text{if } 1 \leq s \leq r + 1 \text{ and } r \equiv 0(\text{mod } 8) \\
\frac{r+2}{2} & \text{if } 1 \leq s \leq r + 1 \text{ and } r \equiv 2 \text{ or } 6(\text{mod } 8) \\
\frac{r+4}{2} & \text{if } 1 \leq s \leq r + 1 \text{ and } r \equiv 4(\text{mod } 8) \\
2r & \text{if } s > r + 1.
\end{cases}
\]
Proof. Let $G$ be the corona product of $C_r$ and $P_s$. Let $u_1, u_2, ..., u_r$ and $v_{i,1}, v_{i,2}, ..., v_{i,s}$ be the vertices of $C_r$ and the vertices of the $i^{th}$ copy of $P_s$ $(1 \leq i \leq r)$, respectively. We prove this theorem by considering two cases.

Case 1. $1 \leq s \leq r+1$.

Subcase (i) $r \equiv 1(\text{mod } 8)$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; ...; v_{r-8,j}, v_{r-7,j}, v_{r-6,j}, v_{r-5,j}\} \cup \{v_{i,j}\} (1 \leq j \leq s)$. It is easily verified that the vertices $u_{r-1}, v_{r-1,j}$, $u_3$ and $v_{3,j}$ are monophonic eccentric dominated by a vertex $v_{1,j}$, the vertices $u_4$, $v_{4,j}$, $u_5$ and $v_{5,j}$ are monophonic eccentric dominated by a vertex $v_{2,j}$, the vertices $u_i$, $v_{i,j}$, $u_5$ and $v_{5,j}$ are monophonic eccentric dominated by a vertex $v_{3,j}$, the vertices $u_2$, $v_{2,j}$, $u_6$ and $v_{6,j}$ are monophonic eccentric dominated by a vertex $v_{4,j}$, ... the vertices $u_{r-1}$, $v_{r-1,j}$, $u_t$ and $v_{t,j}$ are monophonic eccentric dominated by a vertex $v_{8,j}$, the vertices $u_i$, $v_{i,j}$, $u_5$ and $v_{5,j}$ are monophonic eccentric dominated by a vertex $v_{7,j}$, the vertices $u_2$, $v_{2,j}$, $u_t$ and $v_{t,j}$ are monophonic eccentric dominated by a vertex $v_{r-7,j}$, the vertices $u_{r-8}$, $v_{r-8,j}$, $u_t$ and $v_{t,j}$ are monophonic eccentric dominated by a vertex $v_{r-6,j}$, the vertices $u_{r-3}$ and $v_{r-3,j}$ are monophonic eccentric dominated by a vertex $v_{r-3,j}$ and the vertices $u_2$, $v_{2,j}$, $u_t$ and $v_{t,j}$ are monophonic eccentric dominated by a vertex $v_{r-3,j}$. It is clear that $S$ is a minimum total monophonic eccentric dominating set of $G$ and so $\gamma_{te}(G) = \frac{r+1}{2}$.

Subcase (ii) $r \equiv 3(\text{mod } 8)$.

If $r = 3$, let $S = \{v_{1,j}, v_{2,j}\}$ and for other values of $r$, let $S = \{v_{1,j}, v_{2,j}; v_{9,j}, v_{10,j}; ...; v_{r-2,j}, v_{r-1,j}\} \cup \{v_{4,j}, v_{12,j}; ...; v_{r-7,j}\} \cup \{v_{7,j}, v_{15,j}; ...; v_{r-6,j}\} (1 \leq j \leq s)$. By an argument similar to Subcase (i), it can be easily seen that $S$ is a minimum total monophonic eccentric dominating set of $G$ and so $\gamma_{te}(G) = \frac{r+1}{2}$.

Subcase (iii) $r \equiv 5(\text{mod } 8)$.

If $r = 5$, let $S = \{v_{1,j}, v_{2,j}, v_{4,j}\}$ and for other values of $r$, let $S = \{v_{1,j}, v_{2,j}; v_{9,j}, v_{10,j}; ...; v_{r-4,j}, v_{r-3,j}\} \cup \{v_{4,j}, v_{12,j}; ...; v_{r-1,j}\} \cup \{v_{7,j}, v_{15,j}; ...; v_{r-6,j}\} (1 \leq j \leq s)$. By an argument similar to Subcase (i), it can be easily seen that $S$ is a minimum total monophonic eccentric dominating set of $G$ and so $\gamma_{te}(G) = \frac{r+1}{2}$.

Subcase (iv) $r \equiv 7(\text{mod } 8)$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; ...; v_{r-6,j}, v_{r-5,j}, v_{r-4,j}, v_{r-3,j}\} (1 \leq j \leq s)$. By an argument similar to Subcase (i), it can be easily seen that $S$ is a minimum total monophonic eccentric dominating set of $G$ and so $\gamma_{te}(G) = \frac{r+1}{2}$.

Subcase (v) $r \equiv 0(\text{mod } 8)$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; ...; v_{r-7,j}, v_{r-6,j}, v_{r-5,j}, v_{r-4,j}\} (1 \leq j \leq s)$. By an argument similar to Subcase (i), it is clear that $S$ is a minimum total monophonic eccentric dominating set of $G$ and so $\gamma_{te}(G) = \frac{r}{2}$.

Subcase (vi) $r \equiv 2$ or $6(\text{mod } 8)$.

If $r \equiv 2 (\text{mod } 8)$, let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; ...; v_{r-9,j}, v_{r-8,j}, v_{r-7,j}, v_{r-6,j}\} \cup$
\{v_{1,j},v_{i,j}\} (1 \leq j \leq s). By an argument similar to Subcase (i), it is clear that \(S\) is a minimum total monophonic eccentric dominating set of \(G\) and so \(\gamma_{te}(G) = \frac{r+2}{2}\).

If \(r \equiv 6 \pmod{8}\), let \(S = \{v_{1,j},v_{2,j},v_{3,j},v_{4,j}; v_{5,j},v_{10,j},v_{11,j},v_{12,j}; \ldots; v_{r-5,j},v_{r-4,j},v_{r-3,j},v_{r-2,j}\} (1 \leq j \leq s)\). By an argument similar to Subcase (i), it can be easily seen that \(S\) is a minimum total monophonic eccentric dominating set of \(G\) and so \(\gamma_{te}(G) = \frac{r+2}{2}\).

**Subcase (vii)** \(r \equiv 4 \pmod{8}\).

Let \(S = \{v_{1,j},v_{2,j},v_{3,j},v_{4,j}; v_{5,j},v_{10,j},v_{11,j},v_{12,j}; \ldots; v_{r-3,j},v_{r-2,j},v_{r-1,j}\} (1 \leq j \leq s)\). By an argument similar to Subcase (i), it is clear that \(S\) is a minimum total monophonic eccentric dominating set of \(G\) and so \(\gamma_{te}(G) = \frac{r+4}{2}\).

**Case 2.** \(s > r + 1\).

Let \(S = \{v_{1,1},v_{2,1},\ldots,v_{r,1}; v_{1,s},v_{2,s},\ldots,v_{r,s}\}\). In \(r+1 \leq s < 2r\), the vertices \(u_{r-1},v_{r-1,k},u_3,v_{3,k} (s-r+1 \leq k \leq r)\) and \(v_{1,l} (r+1 \leq l \leq s)\) are monophonic eccentric dominated by the vertex \(v_{1,l}\), the vertices \(u_4,v_{4,k},u_l,v_{r,k} (s-r+1 \leq k \leq r)\) and \(v_{2,l} (r+1 \leq l \leq s)\) are monophonic eccentric dominated by the vertex \(v_{2,l}\), \ldots, the vertices \(u_{r-2},v_{r-2,k},u_2,v_{2,k} (s-r+1 \leq k \leq r)\) and \(v_{r,l} (r+1 \leq l \leq s)\) are monophonic eccentric dominated by the vertex \(v_{r,l}\). Also, the vertices \(v_{1,l} (1 \leq l \leq s-r)\) are monophonic eccentric dominated by the vertex \(v_{1,s}\), the vertices \(v_{2,l} (1 \leq l \leq s-r)\) are monophonic eccentric dominated by the vertex \(v_{2,s}\), \ldots, the vertices \(v_{r,l} (1 \leq l \leq s-r)\) are monophonic eccentric dominated by the vertex \(v_{r,s}\).

In \(s \geq 2r\), the vertices \(u_{r-1},u_3\) and \(v_{1,l} \left(\left\lfloor \frac{s+3}{2} \right\rfloor \leq l \leq s\right)\) are monophonic eccentric dominated by the vertex \(v_{1,l}\), the vertices \(u_4,u_l\) and \(v_{2,l} \left(\left\lfloor \frac{s+3}{2} \right\rfloor \leq l \leq s\right)\) are monophonic eccentric dominated by the vertex \(v_{2,l}\), \ldots, the vertices \(u_{r-2},u_2\) and \(v_{r,l} \left(\left\lfloor \frac{s+3}{2} \right\rfloor \leq l \leq s\right)\) are monophonic eccentric dominated by the vertex \(v_{r,l}\). Also, the vertices \(v_{1,l} (1 \leq l \leq \left\lfloor \frac{s+1}{2} \right\rfloor)\) are monophonic eccentric dominated by the vertex \(v_{1,s}\), the vertices \(v_{2,l} (1 \leq l \leq \left\lfloor \frac{s+1}{2} \right\rfloor)\) are monophonic eccentric dominated by the vertex \(v_{2,s}\), \ldots, the vertices \(v_{r,l} (1 \leq l \leq \left\lfloor \frac{s+1}{2} \right\rfloor)\) are monophonic eccentric dominated by the vertex \(v_{r,s}\). Hence, it is clear that \(S\) is a minimum total monophonic eccentric dominating set of \(G\) and so \(\gamma_{te}(G) = 2r\).

**Theorem 2.14** If \(G = C_r \circ C_s\), then

\[
\gamma_{te}(G) = \begin{cases} 
\frac{r+1}{2} & \text{if } s \leq r + 2 \text{ and } r \text{ is odd} \\
\frac{r}{2} & \text{if } s \leq r + 2 \text{ and } r \equiv 0 \pmod{8} \\
\frac{r+2}{2} & \text{if } s \leq r + 2 \text{ and } r \equiv 2 \text{ or } 6 \pmod{8} \\
\frac{r+4}{2} & \text{if } s \leq r + 2 \text{ and } r \equiv 4 \pmod{8} \\
\gamma_{te}(C_s) & \text{if } s > r + 2.
\end{cases}
\]

**Proof.** Let \(G\) be the corona product of \(C_r\) and \(C_s\). Let \(u_1,u_2,\ldots,u_s\) be the vertices of \(C_r\) and let \(C_{i,s}: v_{1,i},v_{i,2},\ldots,v_{i,s},v_{i,l}\) be the vertices of the \(i^{\text{th}}\) copy of \(C_s\) \((1 \leq i \leq r)\). We prove this
theorem by considering two cases.

**Case 1.** $s \leq r + 2$.

**Subcase (i)** $r \equiv 1 \pmod{8}$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}; v_{11,j}, v_{12,j}; \ldots; v_{r-8,j}, v_{r-7,j}, v_{r-6,j}, v_{r-5,j}\}$ $\cup \{v_{j}\}$ $(1 \leq j \leq s)$. It is easily verified that the vertices $u_{r-1}, v_{r-1,j}, u_3$ and $v_{3,j}$ are monophonic eccentric dominated by a vertex $v_{1,j}$, the vertices $u_4$, $v_{4,j}$, $u_r$ and $v_{r-3,j}$ are monophonic eccentric dominated by a vertex $v_{2,j}$, the vertices $u_1$, $v_{1,j}$, $u_5$ and $v_{5,j}$ are monophonic eccentric dominated by a vertex $v_{3,j}$, the vertices $u_2$, $v_{2,j}$, $u_6$ and $v_{6,j}$ are monophonic eccentric dominated by a vertex $v_{4,j}$, .... the vertices $u_{r-j}$, $v_{r-1,j}$, $u_{r-6}$ and $v_{r-6,j}$ are monophonic eccentric dominated by a vertex $v_{r-8,j}$, the vertices $u_r$, $v_{r,j}$, $u_{r-5}$ and $v_{r-5,j}$ are monophonic eccentric dominated by a vertex $v_{r-7,j}$, the vertices $u_{r-8}$, $v_{r-8,j}$, $u_{r-4}$ and $v_{r-4,j}$ are monophonic eccentric dominated by a vertex $v_{r-6,j}$, the vertices $u_{r-7}$, $v_{r-7,j}$, $u_{r-3}$ and $v_{r-3,j}$ are monophonic eccentric dominated by a vertex $v_{r-5,j}$ and the vertices $u_2$, $v_{2,j}$, $u_{r-2}$ and $v_{r-2,j}$ are monophonic eccentric dominated by a vertex $v_{r-1,j}$. It is clear that $S$ is a minimum total monophonic eccentric dominating set of $G$ and so $\gamma_{tm}(G) = \frac{r+1}{2}$.

**Subcase (ii)** $r \equiv 3 \pmod{8}$.

If $r = 3$, let $S = \{v_{1,j}, v_{2,j}\}$ and for other values of $r$, let $S = \{v_{1,j}, v_{2,j}, v_{9,j}, v_{10,j}; \ldots; v_{r-2,j}, v_{r-1,j}\} \cup \{v_{4,j}, v_{12,j}, \ldots; v_{r-7,j}\} \cup \{v_{7,j}, v_{15,j}; \ldots; v_{r-4,j}\}$ $(1 \leq j \leq s)$. By an argument similar to Subcase (i), it can be easily seen that $S$ is a minimum total monophonic eccentric dominating set of $G$ and so $\gamma_{tm}(G) = \frac{r+1}{2}$.

**Subcase (iii)** $r \equiv 5 \pmod{8}$.

If $r = 5$, let $S = \{v_{1,j}, v_{2,j}, v_{4,j}\}$ and for other values of $r$, let $S = \{v_{1,j}, v_{2,j}, v_{9,j}, v_{10,j}; \ldots; v_{r-4,j}, v_{r-3,j}\} \cup \{v_{4,j}, v_{12,j}; \ldots; v_{r-1,j}\} \cup \{v_{7,j}, v_{15,j}; \ldots; v_{r-6,j}\}$ $(1 \leq j \leq s)$. By an argument similar to Subcase (i), it can be easily seen that $S$ is a minimum total monophonic eccentric dominating set of $G$ and so $\gamma_{tm}(G) = \frac{r+1}{2}$.

**Subcase (iv)** $r \equiv 7 \pmod{8}$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \ldots; v_{r-6,j}, v_{r-5,j}, v_{r-4,j}, v_{r-3,j}\}$ $(1 \leq j \leq s)$. By an argument similar to Subcase (i), it can be easily seen that $S$ is a minimum total monophonic eccentric dominating set of $G$ and so $\gamma_{tm}(G) = \frac{r+1}{2}$.

**Subcase (v)** $r \equiv 0 \pmod{8}$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \ldots; v_{r-7,j}, v_{r-6,j}, v_{r-5,j}, v_{r-4,j}\}$ $(1 \leq j \leq s)$. By an argument similar to Subcase (i), it is clear that $S$ is a minimum total monophonic eccentric dominating set of $G$ and so $\gamma_{tm}(G) = \frac{r}{2}$.

**Subcase (vi)** $r \equiv 2 \text{ or } 6 \pmod{8}$.
If \( r \equiv 2 \pmod{8} \), let \( S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \ldots; v_{r-9,j}, v_{r-8,j}, v_{r-7,j}, v_{r-6,j}\} \cup \{v_{r-1,j}, v_{r,j}\} \) (1 \( \leq j \leq s \)). By an argument similar to Subcase (i), it is clear that \( S \) is a minimum total monophonic eccentric dominating set of \( G \) and so \( \gamma_{tme}(G) = \frac{r+2}{2} \).

If \( r \equiv 6 \pmod{8} \), let \( S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \ldots; v_{r-5,j}, v_{r-4,j}, v_{r-3,j}, v_{r-2,j}\} \) (1 \( \leq j \leq s \)). By an argument similar to Subcase (i), it can be easily seen that \( S \) is a minimum total monophonic eccentric dominating set of \( G \) and so \( \gamma_{tme}(G) = \frac{r+2}{2} \).

**Subcase (vii) \( r \equiv 4 \pmod{8} \).**

Let \( S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \ldots; v_{r-3,j}, v_{r-2,j}, v_{r-1,j}, v_{r,j}\} \) (1 \( \leq j \leq s \)). By an argument similar to Subcase (i), it is clear that \( S \) is a minimum total monophonic eccentric dominating set of \( G \) and so \( \gamma_{tme}(G) = \frac{r+4}{2} \).

**Case 2.** \( s > r+2 \).

Let \( S_k \) (1 \( \leq k \leq r \)) be a minimum total monophonic eccentric dominating set of \( C_{k,s} \). It is clear that, in \( G \), any vertex in \( C_{k,s} \) (1 \( \leq k \leq r \)) is monophonic eccentric dominated by a vertex in \( S_k \). Also, the vertices \( u_1, u_2, \ldots, u_r \) are monophonic eccentric dominated by a vertex in \( S_k \) (1 \( \leq k \leq r \)) and hence \( S = \bigcup_{k=1}^{r} S_k \) is a minimum total monophonic eccentric dominating set of \( G \). Thus \( \gamma_{tme}(G) = |S| = r \gamma_{tme}(C_s) \).

**Theorem 2.15** If \( G = C_r \circ W_s \), then

\[
\gamma_{tme}(G) = \begin{cases} 
\frac{r+1}{2} & \text{if } s \leq r+3 \text{ and } r \text{ is odd} \\
\frac{r}{2} & \text{if } s \leq r+3 \text{ and } r \equiv 0 \pmod{8} \\
\frac{r+2}{2} & \text{if } s \leq r+3 \text{ and } r \equiv 2 \text{ or } 6 \pmod{8} \\
\frac{r+4}{2} & \text{if } s \leq r+3 \text{ and } r \equiv 4 \pmod{8} \\
r \gamma_{tme}(W_s) & \text{if } s > r+3.
\end{cases}
\]

**Proof.**

By Theorem 1.1 and by an argument similar to Theorem 2.14, the required result can be got.

**References**


